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Phase transition from periodic to quasiperiodic behaviour in 4D cellular automata

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Abstract. We have found a phase transition from periodic to quasiperiodic behaviour in four-dimensional cellular automata by mixing rules. As far as we know, this kind of phase transition has not been studied before, and may thus serve as a pilot case.

Rather surprisingly, several examples of global behaviour were found in cellular automata (CA) and coupled-map lattices (CML) some years ago [1, 2], though there were arguments against the occurrence of global non-trivial behaviour under generic conditions in extended non-equilibrium systems governed by local interactions only [3–5]. The main idea behind these arguments were that in such an extended system, regions in space with a slightly different phase will naturally emerge from stochastic perturbations. Since all interactions are local, any 'bubbles' of different phase will eventually grow and destroy the global phase of the system. However, if the dimension of such a discrete system is high enough, one could, on the other hand, expect that the system would be able to approximate the behaviour of any iterative function of a single variable. Working along these lines, Chaté and Manneville found several four- and five-dimensional CA with periodic and quasiperiodic behaviour [1]. Later, a three-dimensional CA with quasiperiodic behaviour was found [2]. In a recent work, Grinstein *et al* [6] claimed that in three dimensions, there cannot exist any CA with periodic behaviour under generic condition.

In this paper, we investigate what happens when the periodic and quasiperiodic rules are mixed in four dimensions. The model is described as follows. Consider a four-dimensional hypercubic lattice of side length L with periodic boundary conditions. On each lattice site i there is a binary, time-dependent variable $\sigma_i(t)$ (also called spin). The magnetization, which is the global quantity of main interest, is defined as the fraction of sites with spin 1, i.e. $m(t) = \frac{1}{N}\sum_i \sigma_i(t)$ where N is the total number of lattice sites. Here all initial configurations were made by letting each site have spin 1 or 0 randomly with equal probability. At each time step, the update of the spin at a site i is determined by the sum of the spins in its von Neumann neighbourhood (the site itself and its nearest neighbouring (nn) sites), $h_i(t) = \sigma_i(t) + \sum_{nn} \sigma_{nn}(t)$. The update rule reads as follows:

$$\sigma_i(t+1) = \begin{cases} 1 & \text{if } h_i(t) \in [a, b] \\ 0 & \text{otherwise} . \end{cases} \tag{1}$$

As found by Chaté and Manneville [1], if $a = 3$ and $b = 8$, the magnetization is periodic while if $a = 4$ and $b = 8$, the magnetization is quasiperiodic in time. Here we mix these

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two different rules by assigning the 3–8 rule to each site with a probability p and assigning the 4–8 rule with probability $1 - p$†. After discarding the first 500 iterations, the return map, $m(t + 1)$ versus $m(t)$, is plotted for several p between 0 and 1 in figure 1. The two axes with the same tick represent $m(t)$ and $m(t + 1)$ while the other one is for the mixing probability. It is easy to see that the system is quasiperiodic in the regime with low mixing parameter p where the 4–8 rule dominates. The return map for the quasiperiodic case is characterized by a circuit (here it is more or less triangular). When the mixing parameter p is high, the system is periodic in the sense that the return map is just three points (period-3 orbit). At $p \approx 0.79$ the system changes from quasiperiodic to periodic behaviour.

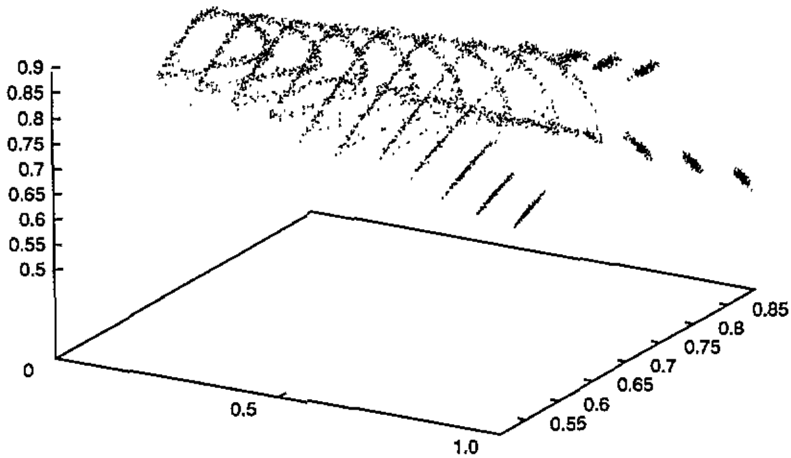

Figure 1. Return maps for different values of mixing parameter p. At $p \approx 0.79$ the behaviour changes from quasiperiodical to periodical.

To characterize this transition, we calculated the Fourier transform of $m(t)$ and chose the absolute value of the component corresponding to the time period $\tau = 3$ as order parameter,

$$\Theta \sim \frac{1}{2\pi T}\left[\left(\sum_{t=1}^{T} m(t)\sin\left(\frac{2\pi t}{3}\right)\right)^2 + \left(\sum_{t=1}^{T} m(t)\cos\left(\frac{2\pi t}{3}\right)\right)^2\right]^{\frac{1}{2}} \tag{2}$$

where T is the total length of the time series used. Figure 2 shows a plot of Θ versus mixing probability p for different lattice sizes. It is clear that there is a transition at $p \approx 0.79$.

To obtain better statistics we averaged over many time series (32 for $L = 16$, 16 for $L = 32$, 4 for $L = 64$) with time length $T = 1024$ for each. The critical transition point is very slightly dependent of the system size, giving rise to the critical exponent ν defined by (see, e.g., [7])

$$p_{\text{eff}} - p_c \sim L^{-1/\nu} \tag{3}$$

where p_{eff} is the critical mixing probability for the phase transition for finite system size (which is the point where the slope $\mathrm{d}\Theta/\mathrm{d}p$ is maximal) and p_c is the critical point for the infinite system. Figure 3 shows a fitting line using the exponent $\nu = 1.2$ for three lattice sizes $L = 16, 32$ and 64. The critical point p_c is found from figure 3 to be about 0.79 ± 0.01. To estimate the exponent β which characterizes how the order parameter varies near p_c, $\Theta \sim (p - p_c)^\beta$, we have done the finite-size scaling using the scaling assumption [7]:

$$\Theta = L^{-\beta/\nu} F[(p - p_c)L^{1/\nu}]. \tag{4}$$

† This mixing rule was employed to be either quenched or annealed, giving no essentially different results.

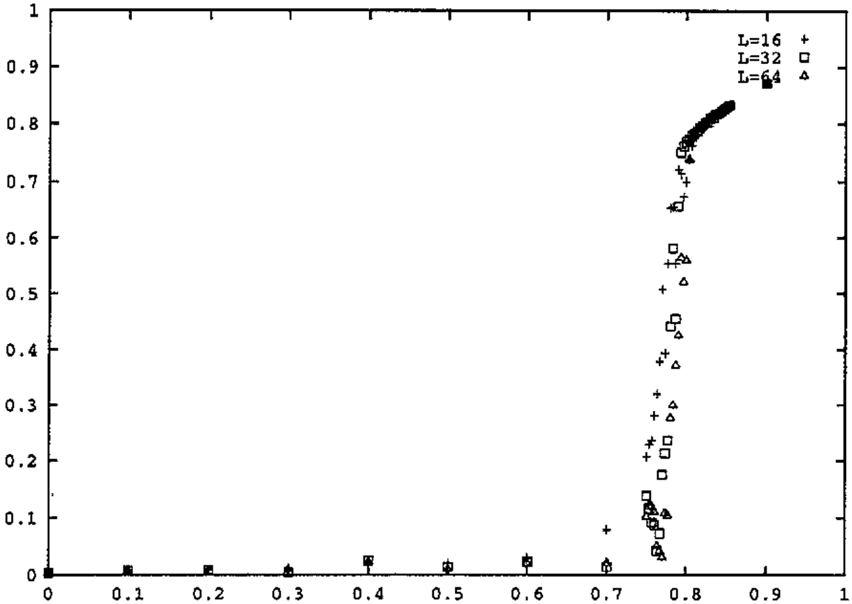

Figure 2. Plot of order parameter versus mixing probability for different lattice sizes.

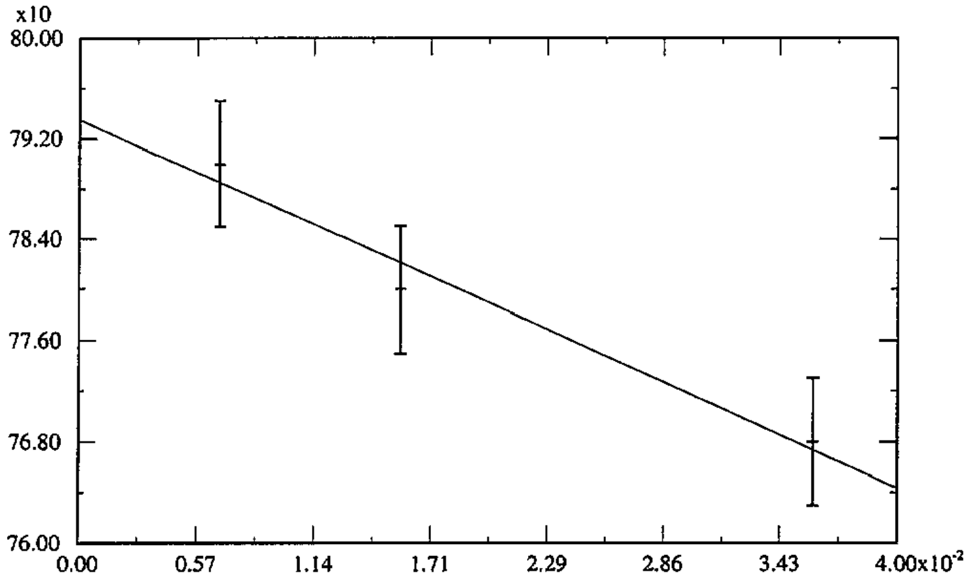

Figure 3. Plot of effective critical point p_{eff} for finite system sizes L versus $L^{-1/\nu}$ with $\nu = 1.2$.

Figure 4 shows data collapse for different system sizes. The horizontal axis represents the quantity $(p - p_c)L^{1/\nu}$ and the vertical axis stands for $\Theta L^{\beta/\nu}$. The fitting exponents and the critical mixing probability p_c used in this figure are $\beta = 0.1 \pm 0.05$, $\nu = 1.1 \pm 0.1$ and $p_c = 0.79 \pm 0.01$, respectively, which are consistent with figure 3.

The simulations were performed using two Connection Machines; a 16K-node CM-2

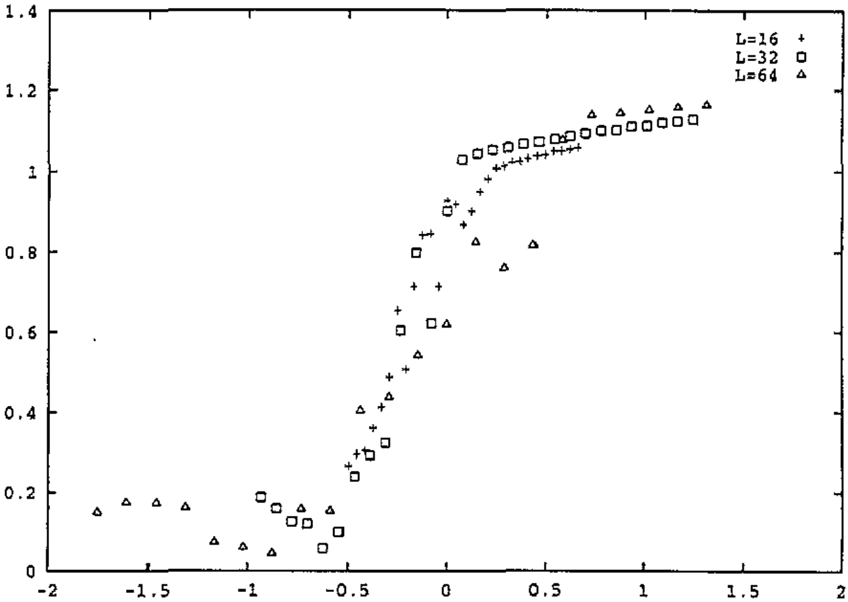

Figure 4. Data collapse for finite-size scaling (equation (4)) using fitting exponents $\beta = 0.1$, $\nu = 1.1$ and the critical mixing probability $p_c = 0.79$.

at GMD in Bonn and an 8K-node CM-200 at PDC in Stockholm. It may be possible to simulate even larger systems if one uses some special techniques, including one linear dimension. With standard programming, however, the 256 Kbits per node available at the 16K CM-2 would only give 16 bits per site for a 128^4 site system, which was not enough for simulating such a system.

In conclusion, we have found a phase transition from periodic to quasiperiodic behaviour in four-dimensional cellular automata by mixing two different rules. The order parameter is chosen to be the absolute value of the Fourier transform of the magnetization corresponding to the time period $\tau = 3$. By simple means, we estimate $p_c \approx 0.79$ and the values of the exponents $\beta \approx 0.1$ and $\nu \approx 1.1$.

Acknowledgments

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